Algorithmic Verification of Linearizability for Ordinary Differential Equations

Dmitry A. Lyakhov[1], Vladimir P. Gerdt[2] and Dominik L. Michels[1,3]

[1]Computational Sciences Group
King Abdullah University of Science and Technology, Thuwal, Saudi Arabia

[2]Laboratory of Information Technologies
Joint Institute for Nuclear Research, Dubna, Russia

[3]Department of Computer Science
Stanford University, Stanford, USA

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Introduction

How to solve differential equation?

$$y''' + \frac{3y'}{y}(y'' - y') - 3y'' + 2y' - y = 0 \qquad (1)$$

It admits rich Lie symmetry group, however Maple solver dsolve outputs

$$y(x) = e^{\int RootOf\left(\int^{\_Z} \frac{2\, RootOf\left(-\left(\int^{\_Z} \frac{1}{4\_h^3 - \_h + 1} d\_h \right) + \int \frac{1}{4\_k^3 - 6\_k^2 + 2\_k - 1} d\_k + \_C1 \right) + 2\_k - 1}{4\_k^3 - 6\_k^2 + 2\_k - 1} d\_k + x + \_C2 \right) dx + \_C3}$$

On the other hand, Eq. (1) admits the linearization [Ibragimov, 2009]

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Prehistory

The linearization problem for a second-order ODE

$$y'' + f(x, y, y') = 0 \tag{2}$$

was solved by Sophus Lie. He showed that only equations of the following form are linearizable by point transformations:

$$f = F_3(x,y)(y')^3 + F_2(x,y)(y')^2 + F_1(x,y)\,y' + F_0(x,y)\,. \tag{3}$$

Theorem.

Equation (2) is linearizable by point transformation if and only if

$$\begin{aligned} &3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} - 3F_1(F_3)_x + 2F_2(F_2)_x \\ &-3F_3(F_1)_x + 3F_0(F_3)_y + 6F_3(F_0)_y - F_2(F_1)_y = 0\,, \\ &(F_2)_{xx} - 2(F_1)_{xy} + 3(F_0)_{yy} - 6F_0(F_3)_x + F_1(F_2)_x \\ &-3F_3(F_0)_x + 3F_0(F_2)_y + 3F_2(F_0)_y - 2F_1(F_1)_y = 0\,. \end{aligned} \tag{4}$$

In this paper we consider ODEs of the form

$$y^{(n)} + f(x, y, y', \ldots, y^{(n-1)}) = 0\,, \quad y^{(k)} := \frac{d^k y}{dx^k} \tag{5}$$

with $f \in \mathcal{C}(x, y, y', \ldots, y^{(n-1)})$ solved with respect to the highest order derivative.

Given an ODE of the form (5), our aim is to check the existence of an invertible transformation

$$u = \phi(x, y)\,, \quad t = \psi(x, y) \tag{6}$$

which maps (5) into a linear n-th order homogeneous equation

$$u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) = 0\,, \quad u^{(k)} := \frac{d^k u}{dt^k}\,. \tag{7}$$

The invertibility of (6) is provided by the inequation

$$J := \phi_x \psi_y - \phi_y \psi_x \neq 0\,.$$

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Lie Symmetry

One way to check the linearizability of Eq. (5) is to follow the classical approach by Lie to study the symmetry properties of Eq. (5) under one-parameter group of transformation [Lie, 1883]

Definition.

Set of transformation $T_a : \tilde{x} = \Phi(x, y, a), \tilde{y} = \Psi(x, y, a)$ is called one-parameter group of transformation of differential equation

$$F(x, y, y', ..., y^{(n)}) = 0$$

if
1) it transforms any solution $y(x)$ in old variables (x, y) to solution $\tilde{y}(\tilde{x})$ in new variables $(\tilde{x}, \tilde{y})$,
2) it is a group: $T_a T_b = T_{a+b}$, where (a – group parameter).

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Infinitesimal Transformation

The key point is to study vector field of infinitesimal transformation, which is the first term in Taylor expansion of one-parameter group of transformation

$$\tilde{x} = x + \varepsilon \underline{\xi(x,y)} + \mathcal{O}(\varepsilon^2)\,, \quad \tilde{y} = y + \varepsilon \underline{\eta(x,y)} + \mathcal{O}(\varepsilon^2)\,. \tag{8}$$

Infinitesimal symmetry operators

$$\mathcal{X} := \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y$$

form Lie algebra L under Lie bracket

$$[\mathcal{X}_1, \mathcal{X}_2] = \mathcal{X}_1\mathcal{X}_2 - \mathcal{X}_2\mathcal{X}_1.$$

Sophus Lie showed that Lie algebra of n-dimensional ODE satisfies

- if $n = 1$, then $dim(L) = \infty$
- if $n = 2$, then $dim(L) \leq 8$
- if $n > 2$, then $dim(L) \leq n + 4$

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Basic Theorem

Linear homogeneous n-th order equation (7) with variable coefficients admits the Lie point symmetry group

- $\tilde{t} = t,\ \tilde{u} = u + c_i \cdot v_i(t),\ i = 1...n$
- $\tilde{t} = t,\ \tilde{u} = c_{n+1} \cdot u$

where c_i, c_{n+1} are constants (the group parameters) and $\{v_i(t)\}$ is the fundamental solution of (7).

The symmetry algebra has the n-dimensional abelian Lie subalgebra

$$L_{n+1} := \{\, \mathcal{X}_i := v_i(t)\,\partial_u\ (i = 1, .., n),\ \mathcal{X}_{n+1} := u\,\partial_u \,\}\,. \tag{9}$$

Theorem [Mahomed,Leach, 1991]

A necessary and sufficient condition for the linearization of (5) with $n \geq 3$ via a point transformation is the existence of an abelian n-dimensional subalgebra in symmetry algebra.

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LinearizationTest I

What can we do algorithmically?

- generation of determining equations
- dimension of solution space (by Differential Dimension Polynomial)
- structure constants of Lie algebra [Reid, 1991]

$$\mathcal{X} = \textit{truncated Taylor series} \rightarrow [\mathcal{X}_i, \mathcal{X}_j] = \sum_{k=1}^{m} C_{i,j}^k \mathcal{X}_k \,, \quad 1 \leq i < j \leq m \,.$$

Theorem.

Eq. (5) with $n \geq 2$ is linearizable by a point transformation if and only if one of the following conditions is fulfilled:

1. $n = 2$, $m = 8$;
2. $n \geq 3$, $m = n + 4$;
3. $n \geq 3$, $m \in \{n + 1, n + 2\}$ and derived algebra is abelian and has dimension n.

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Algorithm: LinearizationTest I (q)

Input: q, a nonlinear differential equation of form (5)

Output: True, if q is linearizable and `False`, otherwise

```
n := order(q);
DS := DeterminingSystem (q);
IDS := InvolutiveDeterminingSystem (DS);
m := dim(LieSymmetryAlgebra) (IDS);
if n = 1 ∨ (n = 2 ∧ m = 8) ∨ (n > 2 ∧ m = n + 4) then
    return True;
elif n > 2 ∧ (m = n + 1 ∨ m = n + 2) then
    L := LieSymmetryAlgebra (IDS);
    DA := DerivedAlgebra (L);
    if DA is abelian and dim(DA) = n then
        return True;
    fi
fi
return False;
```

Differential Thomas Decomposition

The differential Thomas decomposition is universal algorithmic tool, which provides a characteristic decomposition of the radical of the differential ideal, generated by differential system.

Definition.

A differential system is a system $S := \{S^=, S^{\neq}\}$ of differential equations and (possibly) inequations of the form

$$S^= := \{g_1 = 0, \dots, g_s = 0\},\ S^{\neq} := \{h_1 \neq 0, \dots, h_t \neq 0\}, s \geq 1, t \geq 0.$$

The Thomas decomposition [Bachler,Gerdt,Lange-Hegermann,Robertz, 2012] applied to a differential system S yields a finite set of involutive and simple differential systems:

1. every simple system has a solution under C
2. solution spaces of two different systems are distinct

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LinearizationTest II

Substitution

$$[u = \phi(x, y), \quad t = \psi(x, y)] \to u^{(n)}(t) + \sum_k a_k(t)\, u^{(k)}(t) = 0.$$

By differentiating the equality $u(\psi(x, y(x))) = \phi(x, y(x))$

$$u'(t) = \frac{\phi_x + \phi_y y'}{\psi_x + \psi_y y'}\,,$$

$$u''(t) = \frac{\phi_x\psi_y - \phi_y\psi_x}{(\psi_x + \psi_y y')^3}\, y'' + \frac{(\psi_x + \psi_y y')\left(\phi_{xx} + \phi_{xy} y' + \phi_{yy}(y')^2\right)}{(\psi_x + \psi_y y')^3}\,,$$

$$\vdots$$

$$u^{(n)}(t) = \frac{J}{(\psi_x + \psi_y y')^{n+1}}\, y^{(n)} + \frac{P_n(y', \ldots, y^{(n-1)})}{(\psi_x + \psi_y y')^{2n-1}}\,.$$

LinearizationTest II

Definition.

The differential system made up of the above constructed PDE set $S^{=}$ and of the inequation set $S^{\neq} = \{J \neq 0\}$ will be called linearizing differential system.

Theorem.

Eq. (5) is linearizable via a point transformation (6) if and only if the linearizing differential system is consistent, i.e. has a solution. It is equivalent to statement that result of differential Thomas decomposition algorithm applied to linearizing system is non-empty set.

Remark.

Linearizing differential system for given ODE $(n \geq 2)$ is finite-dimensional.

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Algorithm: LinearizationTest II (q, P, H)

Input: q, a nonlinear differential equation of form (5) of order ≥ 2; P, a set of parameters; H, a set of undetermined functions in (x, y)

Output: Set G of differential systems for functions ϕ and ψ in (6) and (possibly) in elements of P and H if (5) is linearizable, and the empty set, otherwise

$n :=$ order(q);
$G := \emptyset$;
$M :=$ numerator(f); $N :=$ denominator(f);
$J := \phi_x \psi_y - \phi_y \psi_x$;
if $n = 2$ then
  $r := u''(t) = 0$;
  $A := \emptyset$;
else
  $r := u^{(n)}(t) + \sum_{k=0}^{n-3} a_k(t) u^{(k)}(t) = 0$;
  $A := \{a_0, \ldots, a_{n-3}\}$;
fi
$r \xrightarrow{\text{by (6)}} y^{(n)} + \frac{R(y', \ldots, y^{(n-1)})}{J \cdot (\psi_x + \psi_y y')^{(n-2)}} = 0$;
$T := R \cdot N - M \cdot J \cdot (\psi_x + \psi_y y')^{(n-2)} = 0$;
$S^{=} := \{c = 0 \mid c \in \text{coeffs}\,(T, \{y', \ldots, y^{(n-1)}\})\}$;
$S^{=} := S^{=} \cup_{p \in P} \{p_x = 0, p_y = 0\}$;
$S^{=} := S^{=} \cup_{a \in A} \{a_x \psi_y - a_y \psi_x = 0\}$;
$S^{\neq} := \{J \neq 0\}$;
$G :=$ ThomasDecomposition $(S^{=}, S^{\neq})$;
return G;

Examples

1.

$$y''' + \frac{3y'}{y}(y'' - y') - 3y'' + 2y' - y = 0$$

2.

$$y'' + F_3(x,y)(y')^3 + F_2(x,y)(y')^2 + F_1(x,y)\,y' + F_0(x,y) = 0$$

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$$y'' + F_3(x,y)(y')^3 + F_2(x,y)(y')^2 + F_1(x,y)\,y' + F_0(x,y) = 0$$

Conclusions

- For the first time, the problem of the linearization test for a wide class of ordinary differential equation of arbitrary order was algorithmically solved.
- LinearizationTest I is a efficient way to check the linearizability of ODE, based only on algorithmic symmetry properties.
- LinearizationTest II allows to check linearizability and to construct linearizing mapping.
- The second algorithm may also improve the built-in Maple solver dsolve of differential equations.
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Acknowledgments

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