Improving cuts by means of lifting

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IMO Seminar, Summer 2014

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- Let $P = (p_1, \dots, p_\ell) \in \mathbb{R}^{n \times \ell}$
- Then *mixed-integer set* of (R, P) with respect to f:

$$X_f(R,P) := \left\{ (s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ \ : \ f + \sum_{i=1}^{k} s_i r_i + \sum_{j=1}^{\ell} y_j p_j \in \mathbb{Z}^n \right\}.$$

Motivation for $X_f(R, P)$

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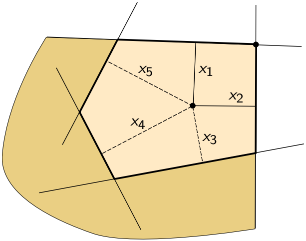

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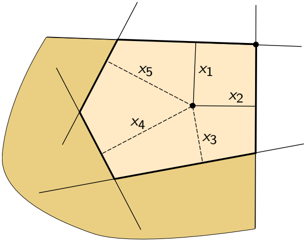

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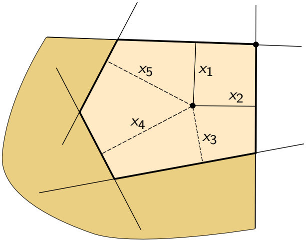

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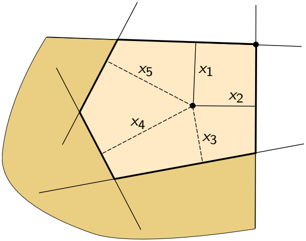

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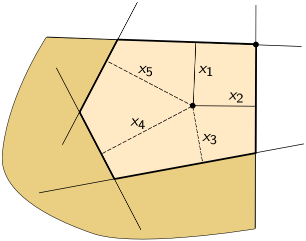

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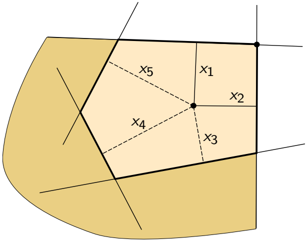
- The variables s_i are continuous non-basic variables.
- The variables y_j are integral non-basic variables.
- The components of $z = f + \sum_{i=1}^k s_i r_i + \sum_{j=1}^\ell y_j p_j$ are basic integral variables.



Cut-generating pairs

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Generation of valid linear inequalities for $X_f(R,P)$.

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Definition: cut-generating pair (Gomory, Johnson)

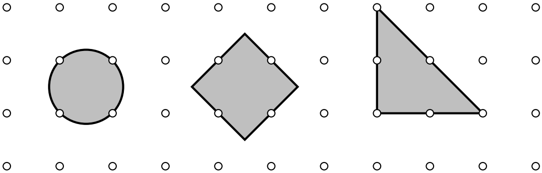
- Let $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$
- (ψ,π) is called a *cut-generating pair* for f if for every choice of (R,P) one has

$$\sum_{i=1}^k \psi(r_i) s_i + \sum_{j=1}^\ell \pi(p_j) y_j \geq 1 \qquad \forall (s,y) \in X_f(R,P)$$

Definition: lattice-free set

A subset B of $\mathbb{R}^n$ is called *lattice-free* (*lf*) if:

- $\mathrm{int}(B) \cap \mathbb{Z}^n = \emptyset$
- B is convex, closed and n-dimensional.



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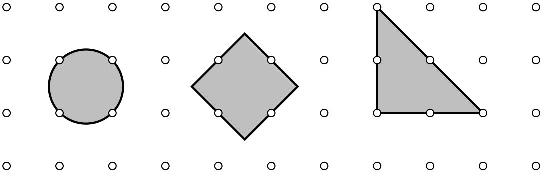


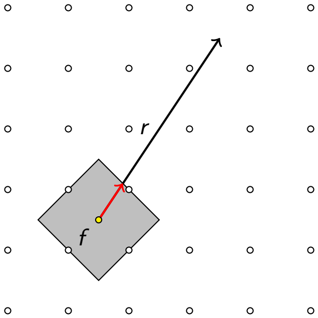
Definition: maximal lattice-free set

A lattice-free set B is called *maximal lattice-free* (*max-lf*) if B is not a subset of any strictly larger lattice-free set.

Definition: gauge function

The *gauge function* of $B - f$ for $f \in \operatorname{int}(B)$:

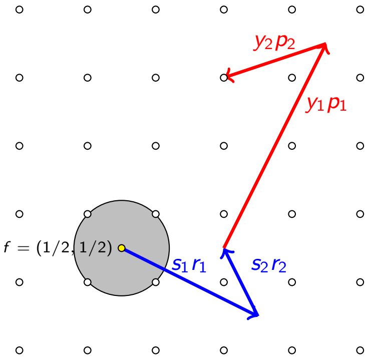
$$\psi_{B-f}(r) := \inf \{\alpha > 0 \ : \ r \in \alpha(B - f)\} .$$



Remark:

Let B be a lf set and let $f \in \text{int}(B)$. Then the pair (ψ, π) with $\psi = \pi = \psi_{B-f}$ is cut-generating.

Example for $\psi = \pi = \psi_{B-f}$



$$f = (1/2, 1/2), \quad R = (r_1, r_2), \quad P = (p_1, p_2), \quad B = \left\{ x \in \mathbb{R}^2 \, : \, \|x - f\| \le \frac{1}{\sqrt{2}} \right\}.$$

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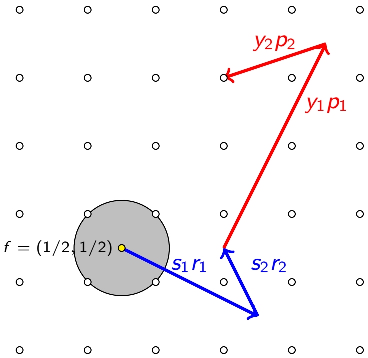

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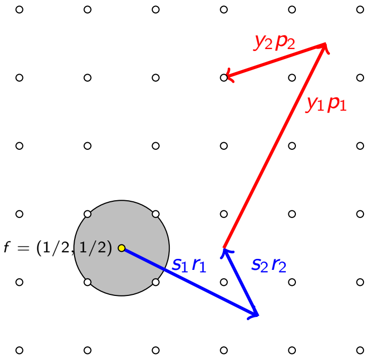

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$$\psi_{B-f}(r_1)s_1 + \psi_{B-f}(r_2)s_2 + \psi_{B-f}(p_1)y_1 + \psi_{B-f}(p_2)y_2 \ge 1.$$

Strengthening cut-generating pairs

Remark:

The cut-generating pair (ψ, π) with $\psi = \pi = \psi_{B-f}$

- is based on the integrality of $z = f + \sum_{i=1}^{k} s_i r_i + \sum_{j=1}^{\ell} y_j p_j$,

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- does not use integrality of $y_1, \ldots, y_\ell$.

The integrality of $y_1, \ldots, y_\ell$ is an important information!

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- In view of the integrality of y:

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- Application of $\psi = \psi_{B-f}$ yields:

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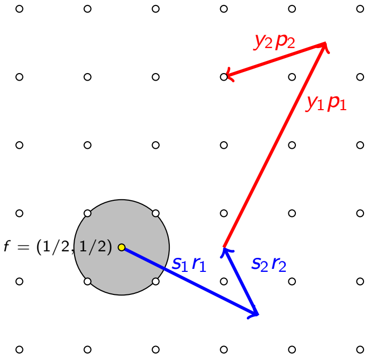
Thus:

Let $\psi = \psi_{B-f}$ and $\pi = \psi^*$ with

$$\psi^*(p) := \inf \left\{ \psi(p + w) \ : \ w \in \mathbb{Z}^n \right\},$$

Then (ψ, π) is a cut-generating pair (a stronger one than (ψ, ψ)).

Example to the lifting technique



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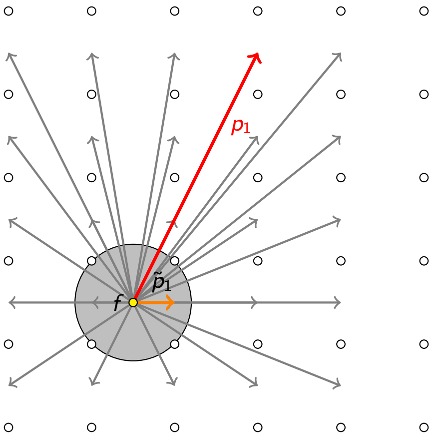

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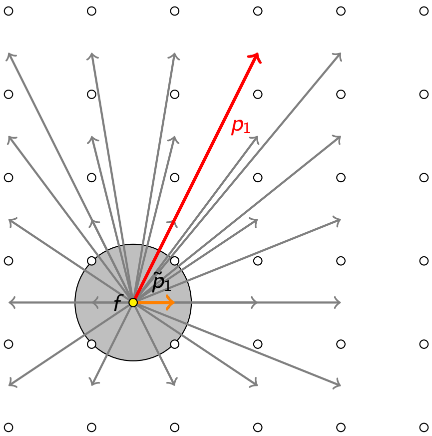

- p_1 can be replaced by a much shorter vector $\tilde{p}_1$.
- Analogously, p_2 can be replaced by a much shorter vector $\tilde{p}_2$.

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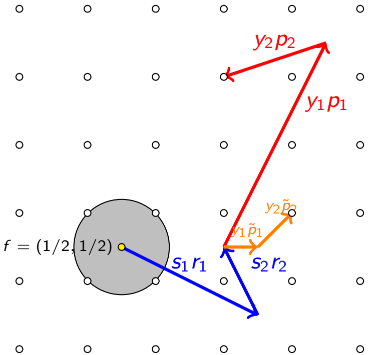

Conclusion:

The inequality

$$s_1\psi(r_1) + s_2\psi(r_2) + y_1\psi(\mathbf{p_1}) + y_2\psi(\mathbf{p_2}) \geq 1 \qquad \forall (s,y) \in X_f(R,P)$$

can be replaced by the much stronger inequality

$$s_1\psi(r_1) + s_2\psi(r_2) + y_1\psi(\tilde{\mathbf{p}}_1) + y_2\psi(\tilde{\mathbf{p}}_2) \geq 1 \qquad \forall (s,y) \in X_f(R,P).$$

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For two liftings π', π'' of ψ, the lifting π' is said to *dominate* π'' if

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Definition: minimal lifting

A lilting π of ψ is called a *minimal lifting* of ψ if π is not dominated by any other lifting π' $(\pi' \neq \pi)$ of ψ.

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- Every gauge function ψ has at least one minimal lifting.
- Generally, ψ can have infinitely many minimal liftings!

A general aim

Aim:

Given $\psi = \psi_{B-f}$, describe the minimal liftings of ψ in a way suitable for algorithmic applications.

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In what follows, let B be a max-lf polytope (polytope = bounded polyhedron).

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Motivation:

Only one best lifting + good formulas for this lifting.

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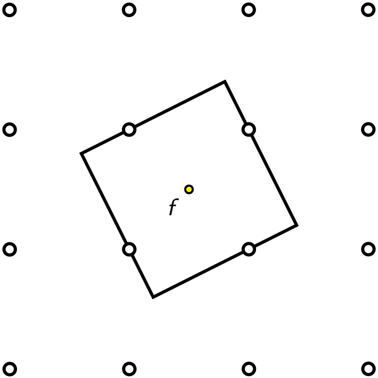
- We call

$$R(B, f) = \bigcup_{F \in \mathcal{F}(B) \setminus \{\emptyset, B\}} \bigcup_{z \in F \cap \mathbb{Z}^n} S_{F,z}(f)$$

  the *lifting region* of (B, f)

Example to the definition fo the lifting region $R(B, f)$



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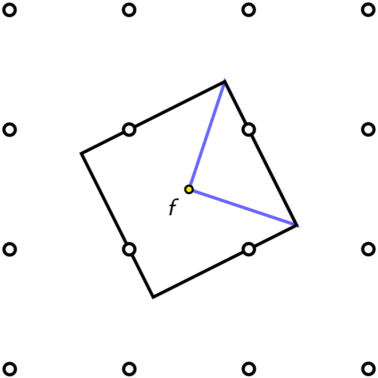

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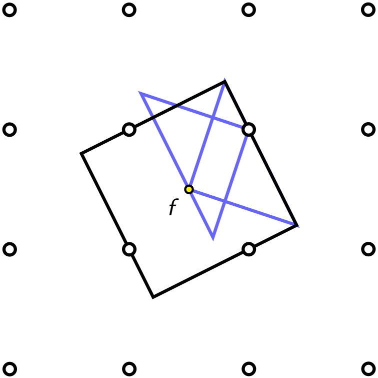

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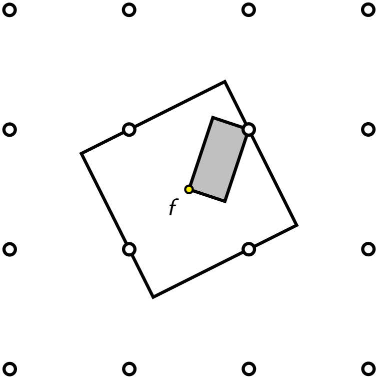

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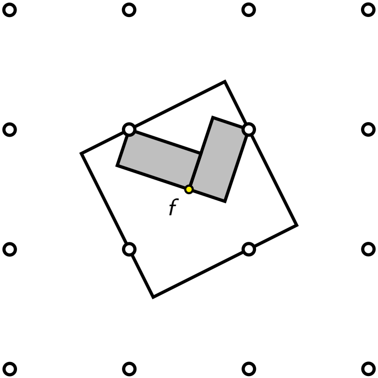

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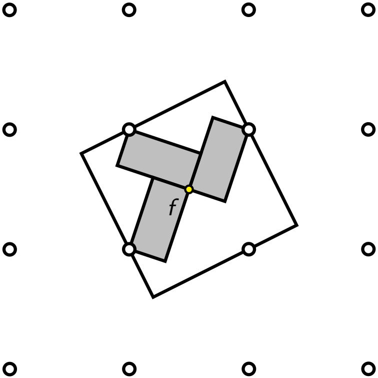

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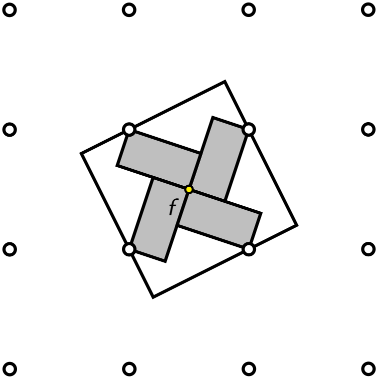

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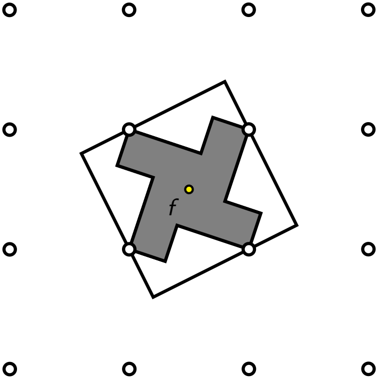

Theorem (Basu, Campêlo, Conforti, Cornuéjols, Zambelli 2013):

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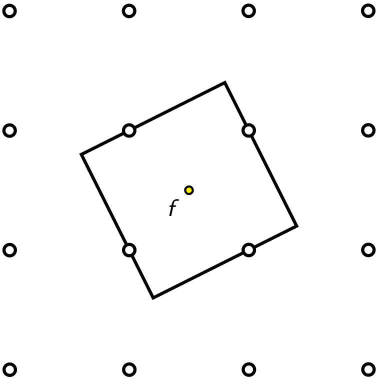

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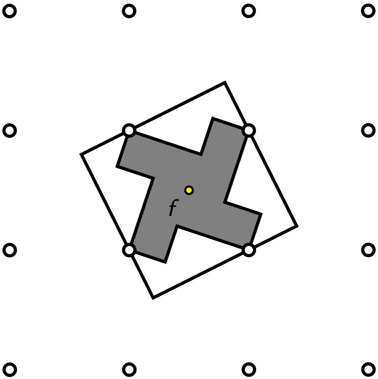

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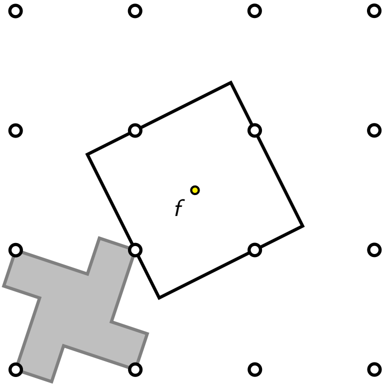

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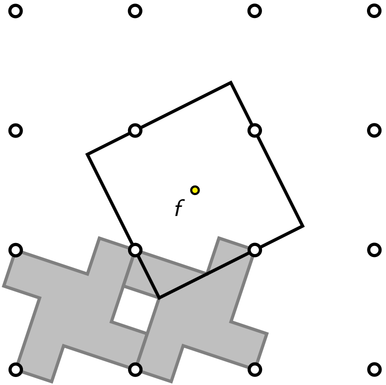

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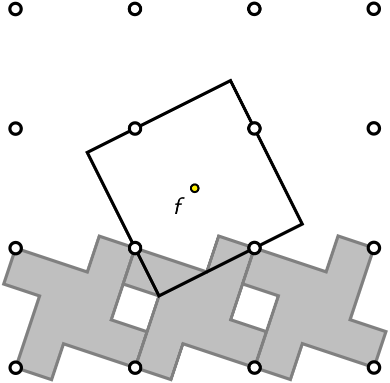

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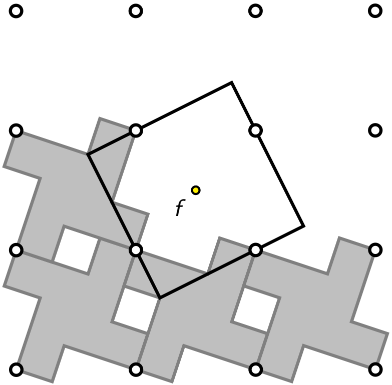

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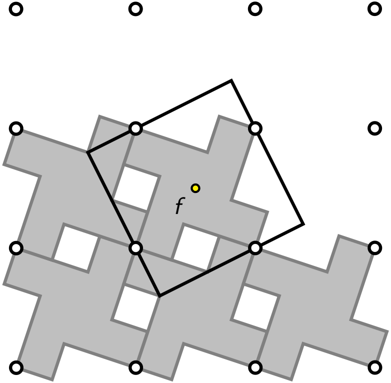

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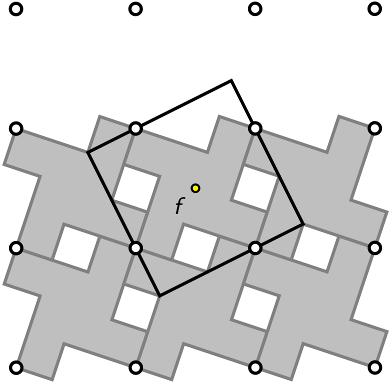

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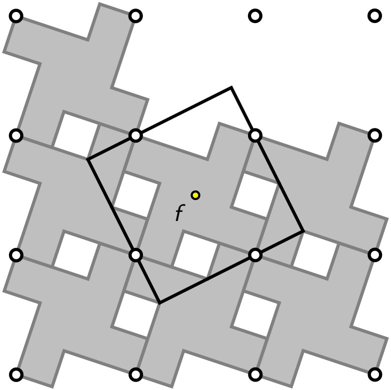

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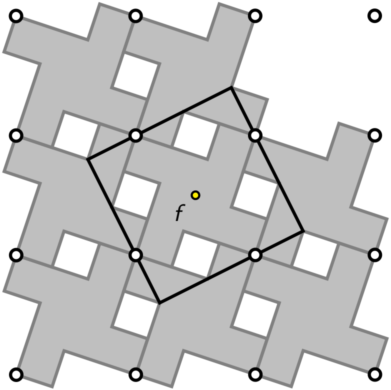

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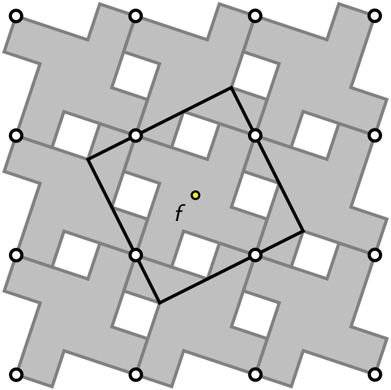

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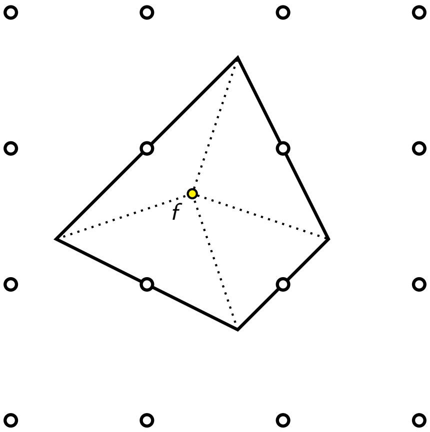

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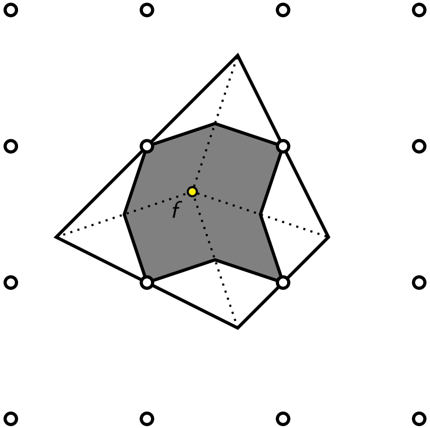

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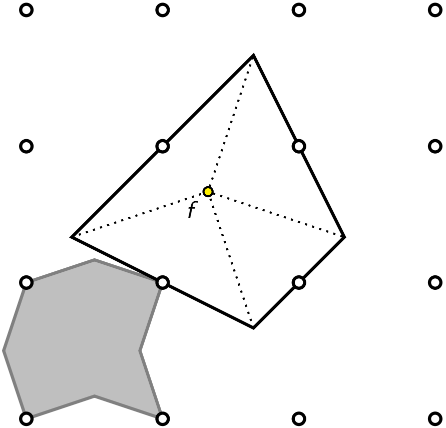

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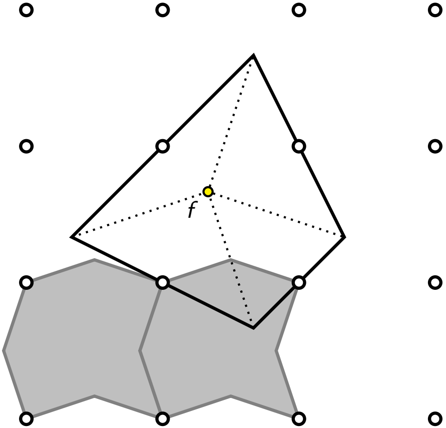

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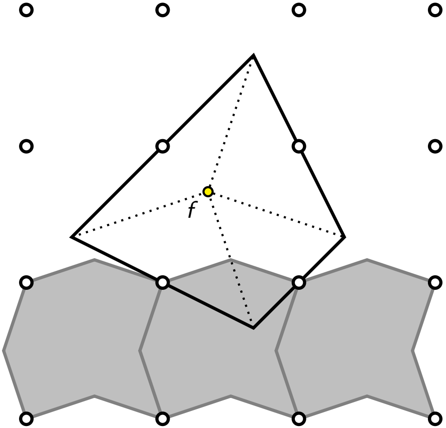

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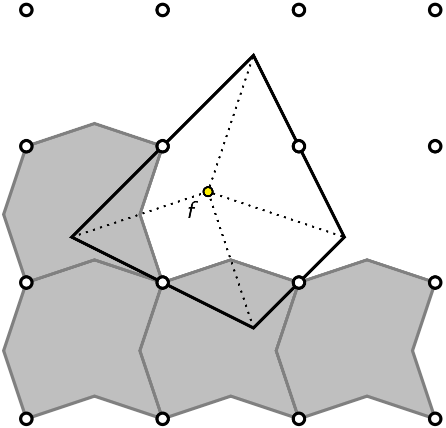

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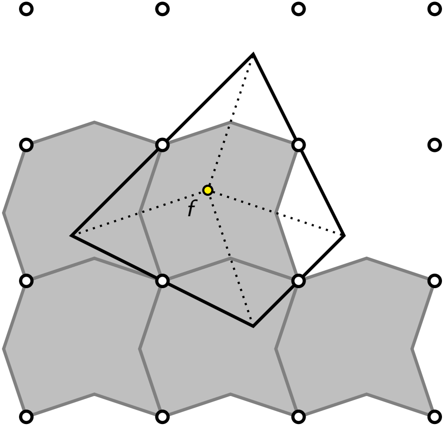

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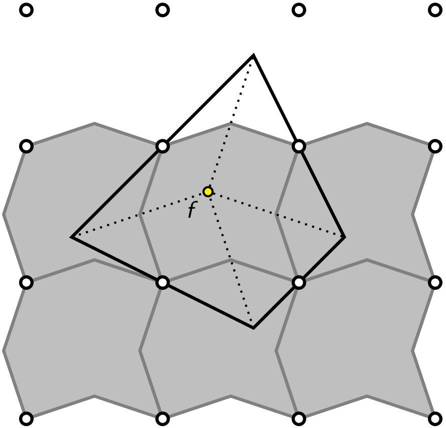

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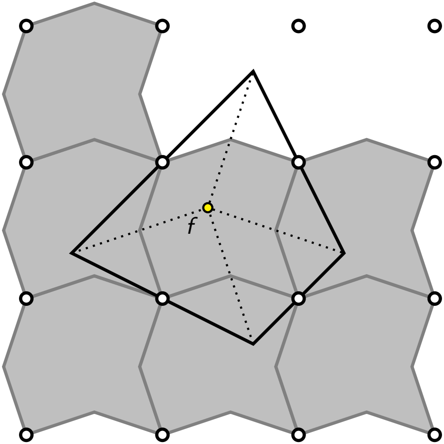

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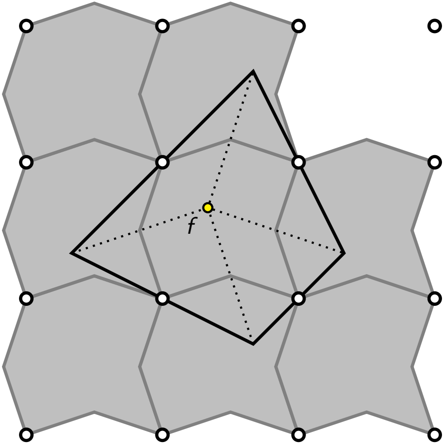

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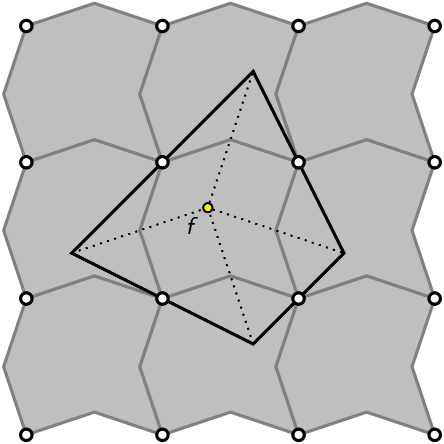

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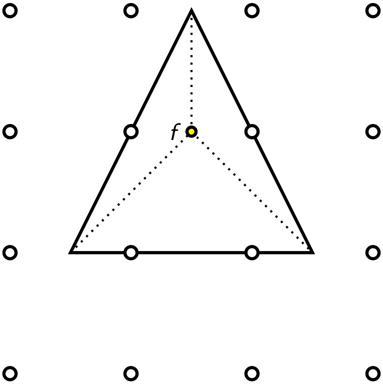

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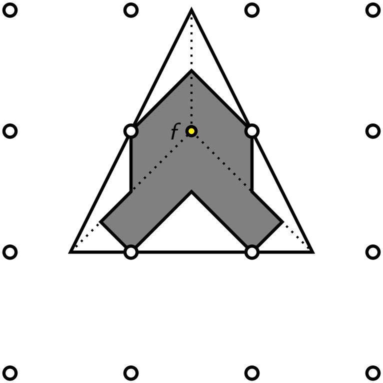

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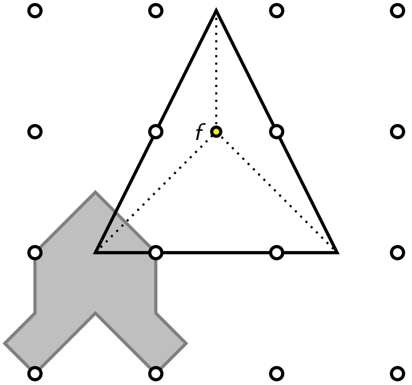

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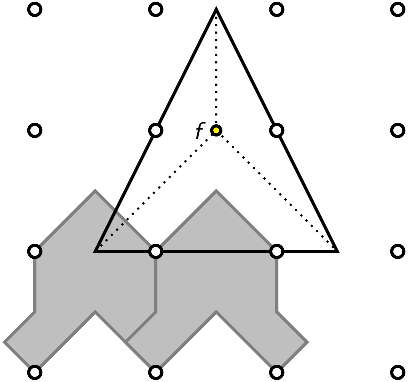

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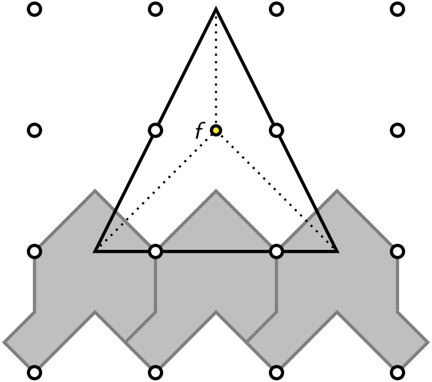

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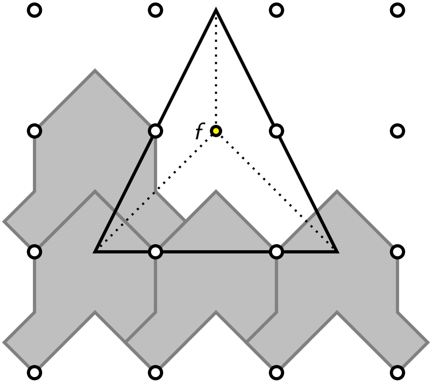

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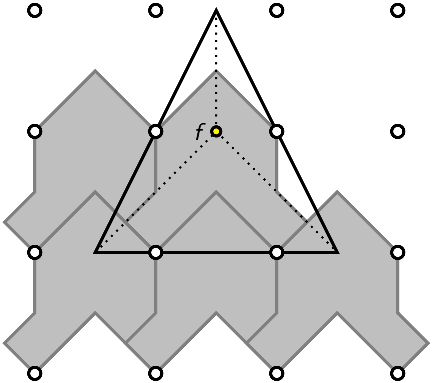

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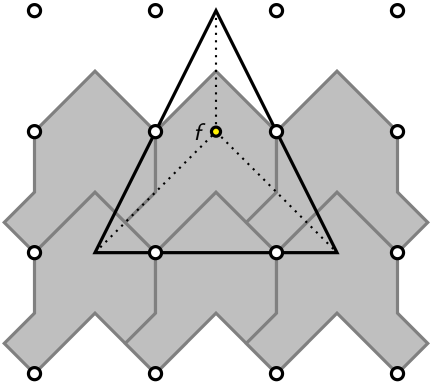

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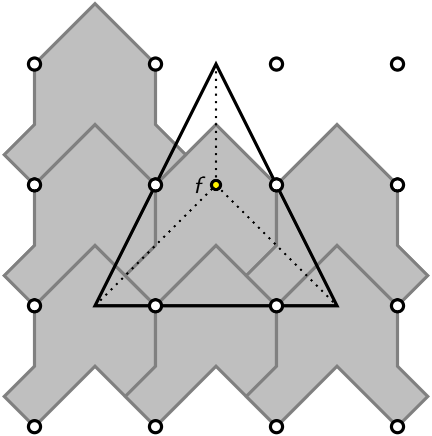

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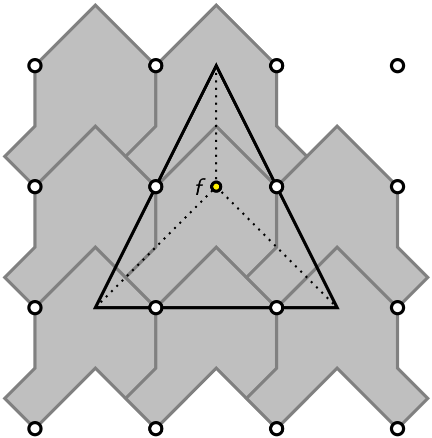

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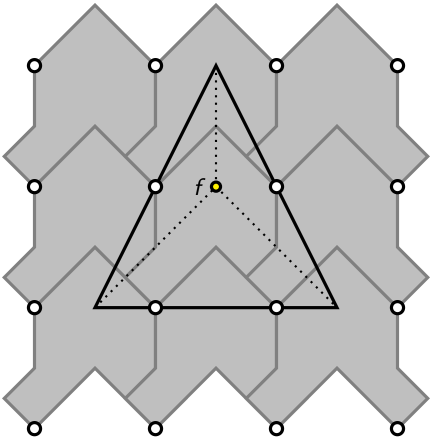

Invariance theorem for simplicial polytopes

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Question (Basu, Cornuéjols, Köppe):

Does the invariance theorem also hold without the simpliciality assumption on B?

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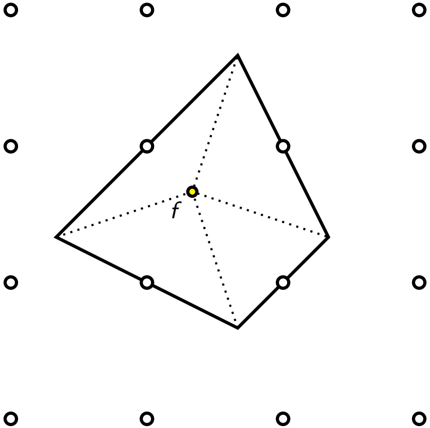
Conclusion:

'Unique minimal lifting' is a property of B alone: the choice of $f \in \operatorname{int}(B)$ plays no role.

Example with a unique minimal lifting



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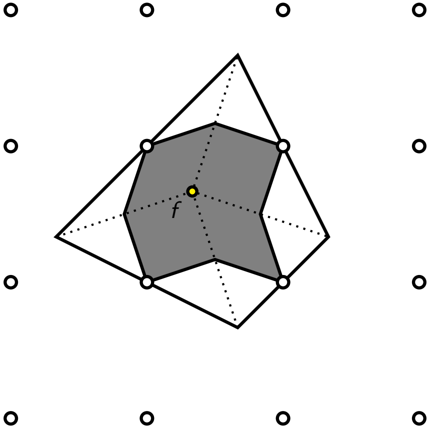

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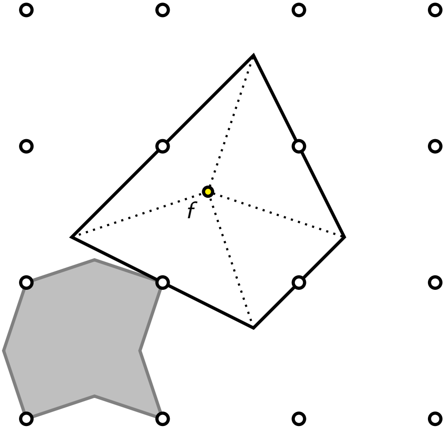

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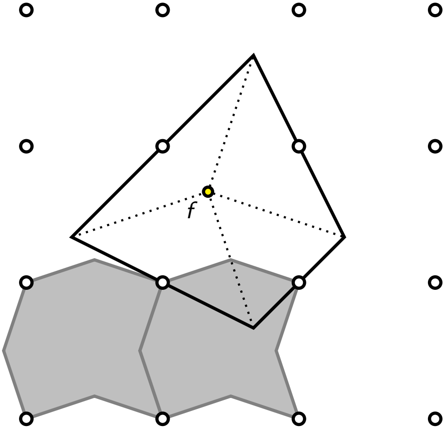

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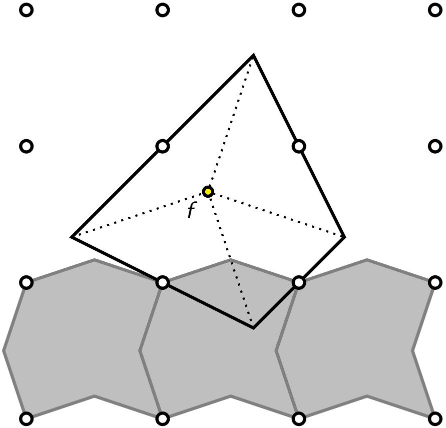

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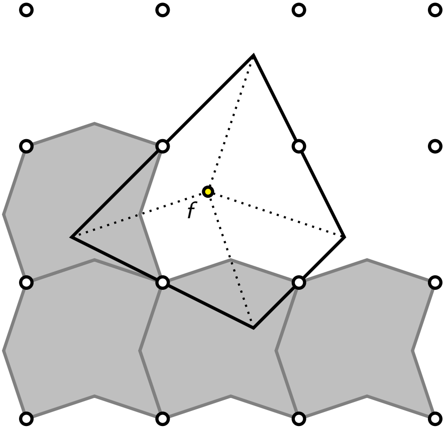

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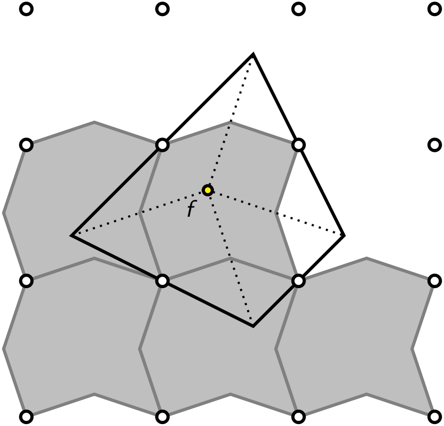

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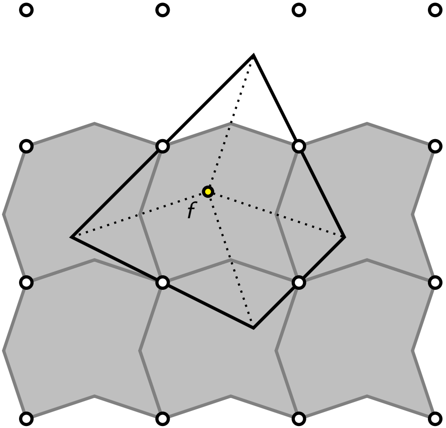

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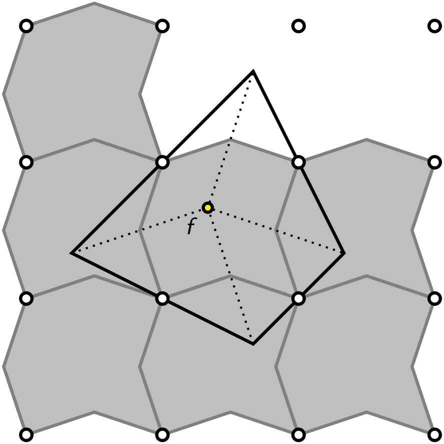

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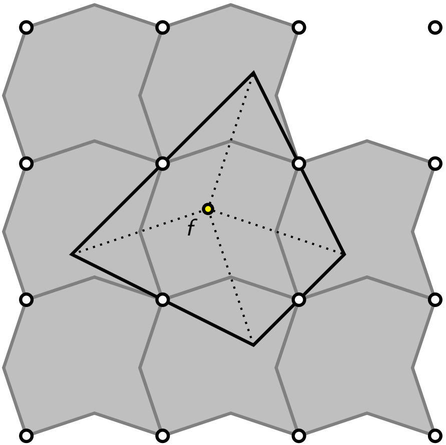

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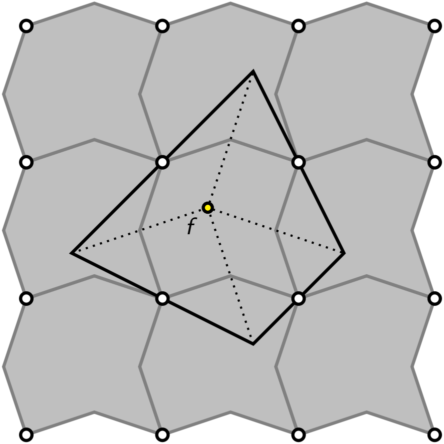
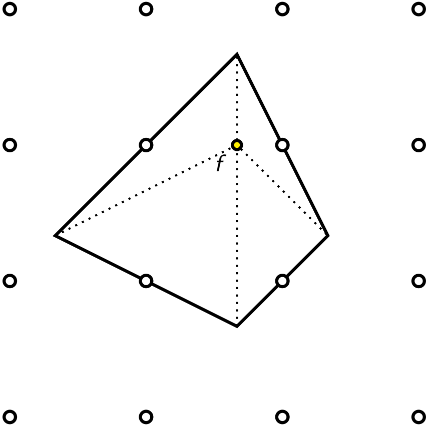


Example with the same B and another $f \in \operatorname{int}(B)$



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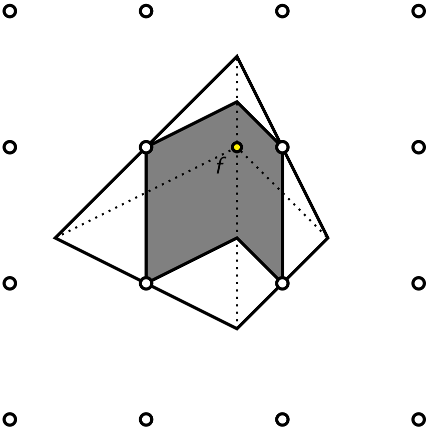

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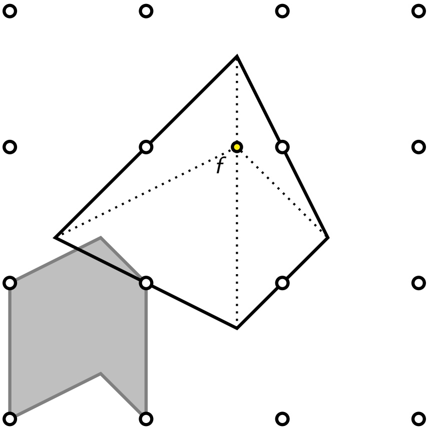

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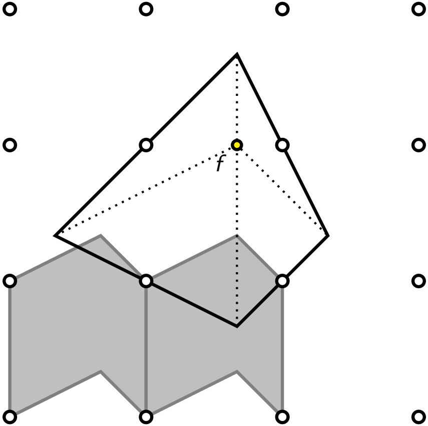

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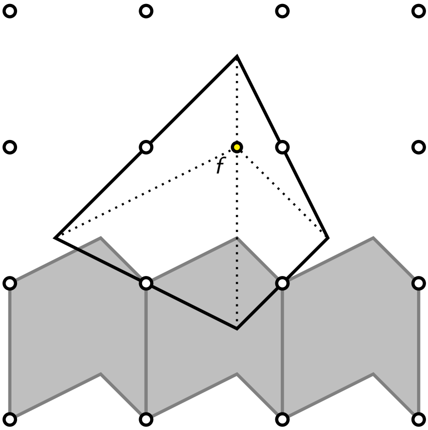

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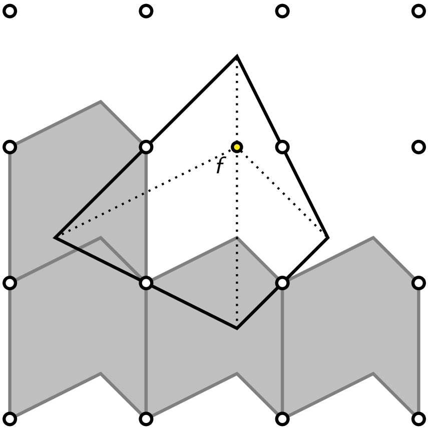

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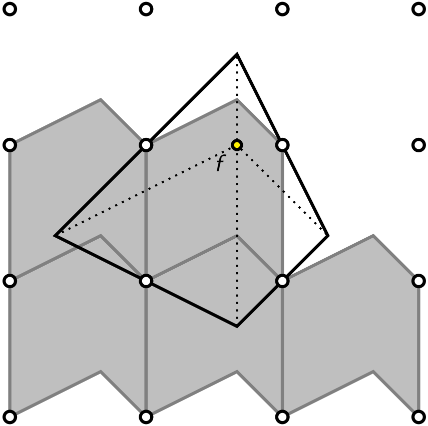

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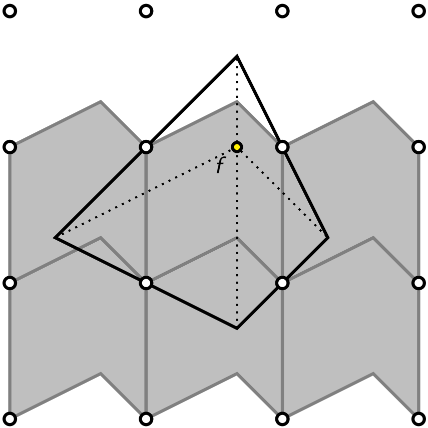

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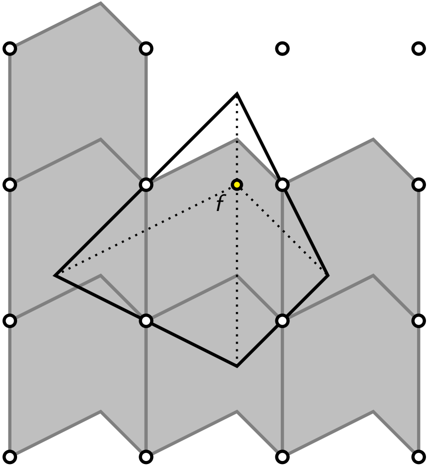

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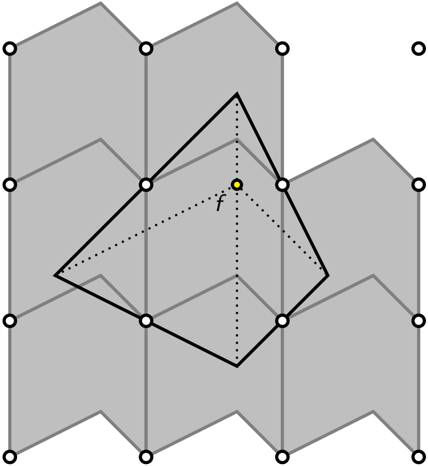

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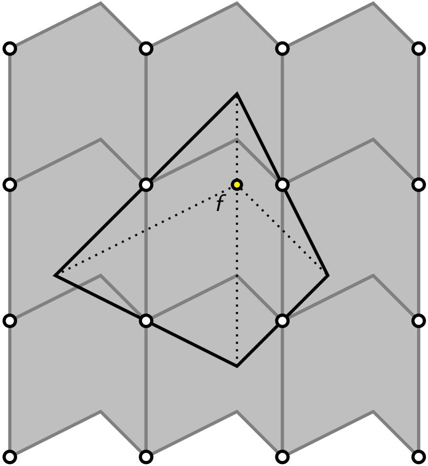

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'Unique minimal lifting', reformulated

Torus

- We introduce the n-dimensionalen torus:

$$\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$$

- Let $\mathrm{vol}_{\mathbb{T}^n}$ be the volume (= Haar measure) on $\mathbb{T}^n$ with $\mathrm{vol}_{\mathbb{T}^n}(\mathbb{T}^n) = 1$.

Remark:

The following conditions are equivalent:

- B has a unique minimal lifting with respect to f
- $R(B, f) + \mathbb{Z}^n = \mathbb{R}^n$
- $R(B, f)/\mathbb{Z}^n = \mathbb{T}^n$
- $\mathrm{vol}_{\mathbb{T}^n}(R(B, f)/\mathbb{Z}^n) = 1$

Torus theorem (A., Basu 2014+):

Let B be a max-lf polytope in $\mathbb{R}^n$. Then the function

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- In the case $F = B$, the set B has a unique minimal lifting for every $f \in \mathrm{int}(B)$.
- In the case $F \neq B$, one has $\mathrm{int}(B) \cap F = \emptyset$ und so B has no unique minimal lifting for every $f \in \mathrm{int}(B)$.

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- For $n = 1$, every max-lf polytope has a unique minimal lifting (trivial).
- In dimension $n = 2$, there is a characterization; Dey, Wolsey 2010.
- For dimensions $n \geq 3$ not much is known!

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Remark:

Pyramids and double pyramids are special coproducts.

Coproduct theorem (A., Basu 2014+):

Let $n := n_1 + n_2$ with $n_1, n_2 \in \mathbb{N}$ and let $0 < \mu < 1$. For $i \in \{1, 2\}$ let B_i be an n_i-dimensional polytope in $\mathbb{R}^{n_i}$ and $c_i \in B_i$.

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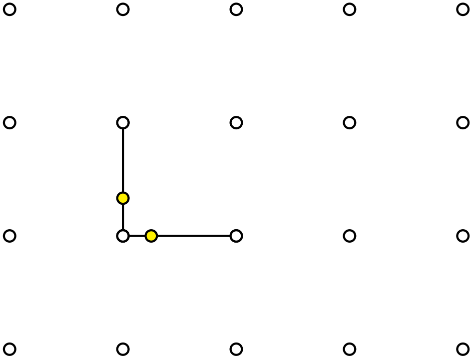
Then the following implication holds:

1. **If B_1, B_2 are max-lf polytopes with a unique minimal lifting, then B is also a max-lf polytope with a unique minimal lifting.**

Example to the coproduct theorem for $n_1 = n_2 = 1$



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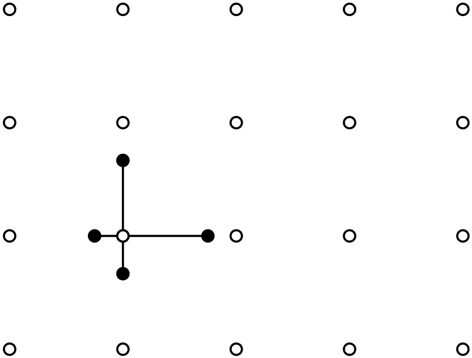

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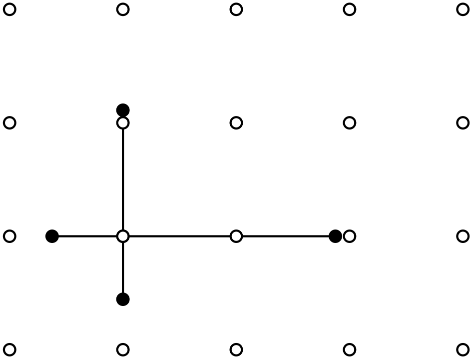

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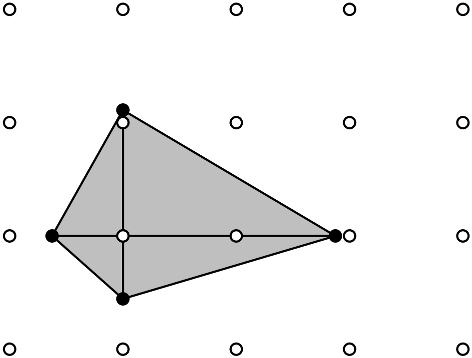

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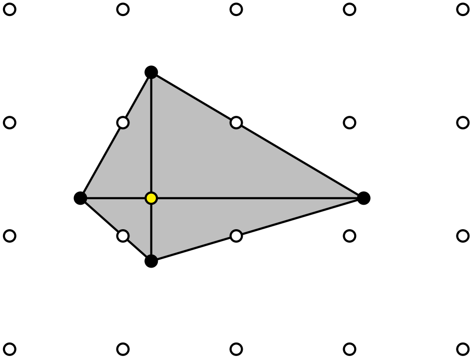

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- In higher dimensions, the coproduct theorem produces **all** max-lf sets with a unique minimal lifting that have been **known so far (and many more).**

Limit argument as a tool

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- Hausdorff distance between two nonempty compact subsets K, L of $\mathbb{R}^n$:

$$\operatorname{dist}(K, L) := \min \{\rho \geq 0 \ : \ K \subseteq L + \mathbb{B}(o, \rho), \ L \subseteq K + \mathbb{B}(o, \rho)\} .$$

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Sei $(B_t)_{t \in \mathbb{N}}$ be a convergent sequence of maximal lattice-free subsets of $\mathbb{R}^n$ whose limit B is a maximal lattice-free set.

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In other words:

The family of all maximal lattice-free subsets of $\mathbb{R}^n$ with a unique minimal lifting is a closed subset of the space of all maximal lattice-free subsets of $\mathbb{R}^n$ (endowed with the Hausdorff metric).

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- **The general belief**: max-lf polytopes B with a unique minimal lifting are rare.
- We would like to confirm this belief, at least in a number of special cases.

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Let B be max-lf **pyramid** in $\mathbb{R}^n$ with a unique minimal lifting such that the relative interior of the basis of B contains **exactly one integral point.**

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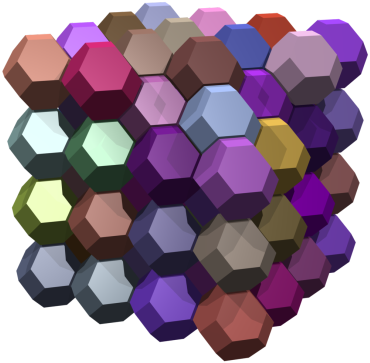

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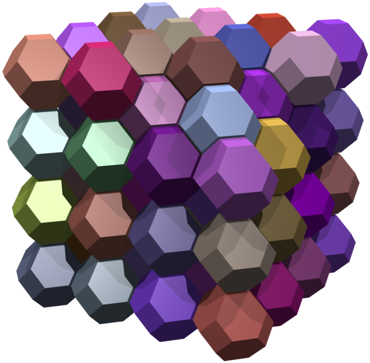

∗ picture borrowed from Wikipedia

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